COURSE NAME: THEORITICAL MECHANICS

COUSRE CODE: MA 278

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INDEX NUMBER: xxxxxxxxxx

ASSIGNMENT 3

1. According to the theory of relativity the mass of a particle is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1-\beta^2}}$$

where v is the speed, m_0 the rest mass, c the speed of light and $\beta = \frac{v}{c}$

a) Show that the time rate of doing work is given by $m_0c^2\frac{d}{dt}(1-\beta^2)^{-\frac{1}{2}}$

b) Deduce from (a) that the kinetic energy is $T = (m - m_0)c^2 = m_0c^2\{(1-\beta^2)^{-\frac{1}{2}} - 1\}$

c) If v is much less than c, show that $T = \frac{1}{2}mv^2$

Answer

a) From Newtons second la;

$$\Rightarrow F = \frac{d}{dt}(mv)$$

m

If; $m = \frac{0}{\sqrt{1-\beta^2}}$

$$\Rightarrow F = \frac{d}{dt}(mv) = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\beta^2}}\right)$$

If W is the work done,

Then

$$\Rightarrow \frac{dW}{dt} = F \,.\, v$$

$$\Rightarrow \frac{dW}{dt} = v.\ \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\beta^2}}\right)$$

$$\Rightarrow \frac{dW}{dt} = m_0 \beta c\ \frac{d}{dt}\left(\frac{\beta c}{\sqrt{1-\beta^2}}\right)$$

$$\Rightarrow \frac{dWd}{dt} = m_0 \beta c^2 \frac{}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right)$$

$$\Rightarrow \frac{dW}{dt} = m_0 c^2\ \frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right)$$

as proved .

b) Since Work done = change in kinetic energy, we have

$\Longrightarrow$ Time rate of doing work = time rate of change in kinetic energy

But from part (a),

$$\frac{dW}{dt} = m_0 c^2 \frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right)$$

$$\Longrightarrow \frac{dW}{dt} = \frac{dT}{dt} = m_0 c^2 \frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right)$$

$$\Longrightarrow \frac{dT}{dt} = m_0 c^2 \frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right)$$

Integrating both sides;

$$\int \frac{dT}{dt}\, dt = \int m_0 c^2 \frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right) dt \quad \Longrightarrow$$

$$\Rightarrow T = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}}\right) + c_1$$

, where $C1$ is constant

- $$\Rightarrow T = \left(\frac{0}{\sqrt{1-\beta 2}}\right)^{2} + {}_{m\,c}C1 ,$$

To find, note that, from the definition $T = 0$ if $v = 0$ or $\beta = 0$

So that we have,

$$0 = \left(\frac{0}{\sqrt{1-(0)^2}}\right) + \, {}^{m\,c}_{2} \Rightarrow C1,$$

$$\Rightarrow 0 = m_0 c^2 + c_1,$$

$$\Rightarrow c_1 = -m_0 c^2$$

Now,

$$\Rightarrow T = \left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right) - m_0 c^2$$

$$\Rightarrow T = m_0 c^2 \left\{ \left(\frac{1}{\sqrt{1-\beta^2}}\right) - 1 \right\}$$

Or

From $T = \left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right) - m_0c^2$

$$T = m_0 c^2 \left\{\left(\frac{1}{\sqrt{1-\beta^2}}\right) - 1\right\}$$

But

$$m = \frac{m_0}{\sqrt{1-\beta^2}}$$

$$\Rightarrow T = (m - m_0)c^2$$

$$\therefore T = m_0c^2\left\{\left(\frac{1}{\sqrt{1-\beta^2}}\right) - 1\right\} = (m - m_0)c^2$$

Hence showed.

c) For $\beta < 1$ we have ,

$$\frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}\beta^2 + \frac{1\cdot 3}{2\cdot 4}\beta^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\beta^6 + \cdots$$

From

m_0c^2

$$T = \left(\frac{}{\sqrt{1-\beta^2}}\right) - m_0c^2$$

$$T = m_0c^2\left[1+\frac{1}{2}\beta^2+\cdots\right] - m_0c^2$$

$$\Rightarrow T = m_0c^2c^2 \; 2\,c \; ^2\left[1+\frac{1}{2}\frac{v}{2}+\cdots\right] - m_0$$

$$T = m_0c^2 + \frac{}{m_0c^2\;1m\;2}\,{}_0cc^2{}^2v^2 -$$

$$\Rightarrow T = \frac{1}{2}mv^2$$

2. From *Figure 2.1* below

   a) show that the Center of Mass (CM) of the uniform thin rod of length ℓ and mass M is at its center.

   b) Determine the CM of the rod assuming its linear mass density λ (its mass per unit length) varies linearly from λ =

$\lambda_0$at the left end to double that value, $\lambda = 2\lambda_0$, at the right end.

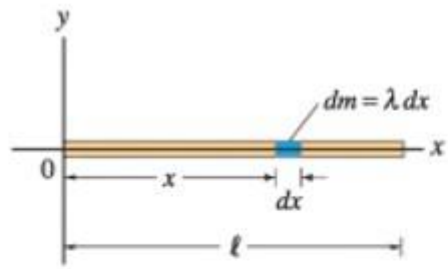

Fig 2.1

Solution

a)

Centre of mass (x_{CM}) $= \frac{1}{M}\int_{x=0}^{\ell} x dm$

But $dm = \lambda\, dx$

$$\Rightarrow \quad x_{CM} = \frac{1}{M}\int_{x=0}^{\ell} x\lambda\, dx$$

$$= \frac{\lambda}{M}\int_{x=0}^{\ell} x\, dx$$

$$= \frac{\lambda}{M}\frac{x^2}{2}\Big|_0^{\ell}$$

$$= \frac{\lambda}{M}\left[\frac{\ell^2}{2} - 0\right]$$

$$= \frac{\lambda}{M}\frac{\ell^2}{2}$$

But $\lambda = \frac{M}{\ell}$

$$\Rightarrow x_{CM} = \frac{1}{M} \cdot \frac{M}{\ell} \cdot \frac{\ell^2}{2}$$

$$\Rightarrow x_{CM} = \frac{\ell}{2}$$

Hence shown.

b) Now we have $\lambda = \lambda_0$ at $x = 0$ and we are told that λ increases linearly to $\lambda = 2\lambda_0$ at $x = \ell$ So we write

$$\lambda = \lambda_0(1 + ax)$$

which satisfies $\lambda = \lambda_0$ at $x = 0$, increases linearly, and gives $\lambda = 2\lambda_0$ at $x = \ell$ if $(1 + a\,\ell) = 2$. In other words, $a = \frac{1}{\ell}$. let $\lambda = \lambda_0 \left(1 + \frac{x}{\ell}\right)$:

$$\Rightarrow x_{CM} = \frac{1}{M}\lambda_0 \int_{x=0}^{\ell} x\left(1 + \frac{x}{\ell}\right) dx$$

$$\Rightarrow x_{CM} = \frac{\lambda_0}{M}\int_{x=0}^{\ell} x\left(1 + \frac{x}{\ell}\right) dx$$

$$= \frac{\lambda_0}{M}\int_{x=0}^{\ell} \left(x + \frac{x^2}{\ell}\right) dx$$

$$= \frac{\lambda_0}{M}\left[\frac{x^2}{2} + \frac{x^3}{3\ell}\right]_0^{\ell}$$

$$= \frac{\lambda_0}{M}\left[\frac{\ell^2}{2} + \frac{\ell^3}{3\ell}\right]$$

$$= \frac{\lambda_0}{M}\left[\frac{\ell^2}{2} + \frac{\ell^2}{3}\right] c$$

$$= \frac{\lambda_0}{M}\frac{5\ell^2}{6}$$

Now let us write M in terms of λ_0 and ℓ . We can write

$$M = \int_{x=0}^{\ell} dm$$

$$= \int_{x=0}^{\ell} \lambda \, dx$$

$$= \lambda_0 \int_{x=0}^{\ell} \left(1 + \frac{x}{\ell}\right) dx$$

$$= \lambda_0 \left[x + \frac{x^2}{2\ell}\right]_0^{\ell}$$

$$= \lambda_0 \left[\ell + \frac{\ell^2}{2\ell}\right]$$

$$= \lambda_0 \left(\ell + \frac{\ell}{2}\right)$$

$$M = \frac{3\lambda_0 \ell}{2}$$

Then

$$\Rightarrow x_{CM} = \frac{\lambda_0}{M}\frac{5\ell^2}{6}$$

$$x_{CM} = \frac{5\ell^2\lambda_0}{6} \cdot \frac{2}{3\lambda_0 \ell}$$

$$x_{CM} = \frac{10\ell}{18}$$

$$x_{CM} = \frac{5\ell}{9}$$

Is greater than halfway along the rod, as we want since there is more mass to the right.